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Approximating the Weight Function for Orthogonal Polynomials on Several Intervals

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A number of methods are available to approximate the weight function for orthogonal polynomials on an interval of the real line. We present some methods to approximate weight functions for orthogonal polynomials on several intervals and give an upper bound for the error in the approximation. We introduce Turán determinants on several intervals and show that these have similar properties as in the one-interval case. These Turán determinants are useful if one deals with sieved orthogonal polynomials on one interval. The proofs depend on asymptotic properties of orthogonal polynomials with asymptotically periodic recurrence coefficients, which are of independent interest.

1. INTRODUCTION

Suppose μ is a positive Borel measure on a compact set of the real line. Then there is a unique sequence of polynomials $\{p_n(x) : n = 0, 1, ...\}$ such that

$$\begin{aligned} \int p_n(x)\, p_m(x)\, d\mu(x) &= \delta_{m,n}, \qquad (m, n \geqslant 0), \\ p_n(x) &= \gamma_n x^n + \cdots, \qquad \gamma_n > 0. \end{aligned} \tag{1.1}$$

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These *orthonormal polynomials* satisfy a simple three term recurrence formula

$$\begin{gathered} xp_n(x) = a_{n+1}\, p_{n+1}(x) + b_n\, p_n(x) + a_n\, p_{n-1}(x), \qquad n \geqslant 0, \\ p_{-1}(x) = 0, \qquad p_0(x) = 1, \end{gathered} \tag{1.2}$$

with $a_n = \gamma_{n-1}/\gamma_n > 0$ and $b_n \in \mathbf{R}$. This can easily be verified by expanding $xp_n(x)$ in a Fourier series using the orthonormal functions $\{p_n(x)\}$; because of (1.1) all but three Fourier coefficients vanish. If the support of μ is compact then the recurrence coefficients a_n and b_n are bounded. Conversely, if $\{p_n(x)\}$ are given by the recurrence formula (1.2) with $a_n > 0$ and $b_n \in \mathbf{R}$, then there exists a positive Borel measure μ such that (1.1) holds and therefore these polynomials are orthonormal polynomials. This is known as Favard's theorem. If a_n and b_n are bounded, then this measure is unique and the support of μ is compact. In many cases the polynomials $p_n(x)$ are given by the recurrence formula (1.2) and then one wants to know the measure μ. If these coefficients are known analytically and if they are well-behaved functions of n (say rational functions of n or rational functions of q^n for some real number q) then one can use analytic techniques (generating functions, differential equations, etc.) to find explicit formulas for μ in a number of cases. In practice, however, the recurrence coefficients may be given numerically and then one is interested in methods to approximate the orthogonality measure μ.

A technique which is often used consists of approximating the Stieltjes transform

$$S(z; \mu) = \int \frac{1}{z - x}\, d\mu(x)$$

using Padé approximants. If one uses a denominator of degree n and a numerator of degree $n-1$ then the denominator polynomial is precisely the orthogonal polynomial $p_n(x)$ and the numerator is the associated polynomial $p_{n-1}^{(1)}(x)$. In general, the *associated polynomials of order* k ($k \geqslant 0$) are given by the shifted recurrence formula

$$xp_n^{(k)}(x) = a_{n+k+1}\, p_{n+1}^{(k)}(x) + b_{n+k}\, p_n^{(k)}(x) + a_{n+k}\, p_{n-1}^{(k)}(x), \qquad n \geqslant 0, \tag{1.3}$$

with initial conditions $p_{-1}^{(k)}(x) = 0$ and $p_0^{(k)}(x) = 1$. By inverting the Stieltjes transform one easily sees that in this method one approximates the measure μ by a purely discrete measure with jumps at the zeros of $p_n(x)$. This is a big disadvantage if it is known that the measure μ is absolutely continuous, in which case there exists a *weight function* $w(x) \geqslant 0$ such that $d\mu(x) = w(x)\, dx$. In such cases it is much more interesting to find an approximation to μ which is itself absolutely continuous or to approximate the weight function $w(x)$. Such approximations are available. One such method uses *Christoffel functions*

$$\lambda_n(x) = \left\{ \sum_{k=0}^{n-1} p_k^2(x) \right\}^{-1},$$

and a good survey that demonstrates their usefulness is given by Nevai [21]. Recently, it was shown that for orthogonal polynomials on $[-1, 1]$ the Szegő condition

$$\int_{-1}^{1} \frac{\log w(x)}{\sqrt{1-x^2}}\, dx > -\infty$$

implies that

$$\lim_{n \to \infty} n\lambda_n(x) = \pi \sqrt{1-x^2}\, w(x)$$

almost everywhere in $[-1, 1]$ (Máté, Nevai, and Totik [18, Theorem 4]). Another method to approximate the weight function is given by Lebedev [13].

In this paper we would like to emphasize approximations using Turán determinants or similar expressions. We define the *Turán determinant* as follows:

$$\begin{aligned} D_n(x) &= p_n^2(x) - \frac{a_{n+1}}{a_n}\, p_{n+1}(x)\, p_{n-1}(x) \\ &= \frac{1}{a_n} \begin{vmatrix} p_n(x) & a_{n+1}\, p_{n+1}(x) \\ p_{n-1}(x) & a_n\, p_n(x) \end{vmatrix}. \end{aligned} \tag{1.4}$$

The name Turán determinant is used because Paul Turán [22] was the first to study such determinants: in particular he proved the inequality

$$P_n^2(x) - P_{n-1}(x) P_{n+1}(x) \geqslant 0, \qquad -1 \leqslant x \leqslant 1$$

for Legendre polynomials, with equality when $x = \pm 1$. We have introduced the factor a_{n+1}/a_n in our definition (1.4) because then the coefficient of x^{2n} vanishes. If the recurrence coefficients converge,

$$\lim_{n \to \infty} a_n = 1/2, \qquad \lim_{n \to \infty} b_n = 0, \tag{1.5}$$

then the following result is known (Máté and Nevai [14, p. 612], Van Assche [25, Theorem 12, p. 454]):

THEOREM 1. *Suppose that* (1.5) *holds and that the recurrence coefficients are of bounded variation, i.e.,*

$$\sum_{k=0}^{\infty} (|b_k - b_{k+1}| + |a_{k+1} - a_{k+2}|) < \infty; \tag{1.6}$$

then μ *is absolutely continuous in* $(-1, 1)$, $w(x) = \mu'(x)$ *is strictly positive and continuous on* $(-1, 1)$, *and*

$$\lim_{n \to \infty} D_n(x) = \frac{2}{\pi} \frac{\sqrt{1-x^2}}{w(x)} \tag{1.7}$$

holds uniformly on every closed subset of $(-1, 1)$. *An upper bound for the error of approximation for* x *on a closed subset of* $(-1, 1)$ *is given by*

$$\left| D_n(x) - \frac{2}{\pi} \frac{\sqrt{1-x^2}}{w(x)} \right| \leqslant C \sum_{k=n+1}^{\infty} (|b_k - b_{k-1}| + |a_k - a_{k-1}|), \tag{1.8}$$

where C *is a constant depending on the closed subset.*

Turán determinants are harder to handle when $x = \pm 1$. One can use expressions similar to Turán determinants to get uniform approximations on the closed interval $[-1, 1]$, still assuming the condition (1.5): for $x \in [-1, 1]$ we set $x = \cos\theta$ and define

$$\psi_n(x) = p_n(x) - 2a_n e^{-i\theta} p_{n-1}(x); \tag{1.9}$$

then

$$\begin{aligned} |\psi_n(x)|^2 &= p_n^2(x) - 4a_n x p_n(x) p_{n-1}(x) + 4a_n^2 p_{n-1}^2(x) \\ &= p_n^2(x) - 4a_n a_{n+1} p_{n+1}(x) p_{n-1}(x) - 4a_n b_n p_n(x) p_{n-1}(x) \end{aligned}$$

and if (1.5) holds then $|\psi_n(x)|^2$ is close to the Turán determinant $D_n(x)$. The following result holds (Nevai [20, Theorem 40, p. 143], Geronimo and Case [6, Appendix B, pp. 487–489], Van Assche [25, Theorem 6, p. 445]):

THEOREM 2. *Suppose that* (1.5) *holds and moreover that*

$$\sum_{k=0}^{\infty} (|1 - 4a_{k+1}^2| + 2|b_k|) < \infty. \tag{1.10}$$

Then μ *is absolutely continuous in* $(-1, 1)$ *and* $w(x) = \mu'(x)$ *is strictly positive and continuous on* $(-1, 1)$. *Furthermore*

$$\lim_{n \to \infty} |\psi_n(x)|^2 = \frac{2}{\pi} \frac{\sqrt{1-x^2}}{w(x)} \tag{1.11}$$

holds uniformly for x *on a closed subset of* $(-1, 1)$. *If the stronger condition*

$$\sum_{k=0}^{\infty} (k+1)(|1-4a_{k+1}^2| + 2|b_k|) < \infty \tag{1.12}$$

holds, then (1.11) *holds uniformly on* $[-1, 1]$ *and one has an upper bound for the error of approximation, given by*

$$\left| |\psi_n(x)| - \prod_{k=n+1}^{\infty} (2a_k) \sqrt{\frac{2}{\pi} \frac{\sqrt{1-x^2}}{w(x)}} \right|$$

$$\leqslant (2^n a_1 a_2 \cdots a_n)^{-1} \sum_{k=n}^{\infty} \frac{3(k+1)}{1+(k+1)\sin\theta} (|1-4a_{k+1}^2| + 2|b_k|)$$

$$\times \exp\left\{ \sum_{k=0}^{\infty} \frac{3(k+1)}{1+(k+1)\sin\theta} (|1-4a_{k+1}^2| + 2|b_k|) \right\}. \tag{1.13}$$

Of course the absolute continuity of μ and the continuity and strict positiveness of μ' already follow from Theorem 1 because the convergence in (1.10) implies the convergence in (1.6). The approximation suggested by Theorem 2 has already been applied succesfully to some weights arising in statistical physics (Van Assche, Turchetti, and Bessis [27]).

In this paper we will generalize Theorem 1 and Theorem 2 for orthogonal polynomials on several intervals. Instead of the condition (1.5) we assume convergence modulo N, where N is a positive integer. This means that we will consider *asymptotically periodic recurrence coefficients.* Orthogonal polynomials with asymptotically periodic recurrence coefficients have been studied by Geronimus [9, 10], Aptekarev [2], Geronimo and Van Assche [7], Van Assche [23, 24], and Grosjean [11]. These polynomials have a measure μ supported on at most N disjoint intervals and in addition μ may have a denumerable number of jumps which can only accumulate on these intervals. The special case when the intervals touch each other leads to *sieved orthogonal polynomials.*

2. Asymptotically Periodic Recurrence Coefficients

Instead of assuming that (1.5) holds, we assume that we are given two periodic sequences $a_{n+1}^0 > 0$ and b_n^0 $(n = 0, 1, ...)$ such that

$$\begin{aligned} a_{n+N}^0 &= a_n^0 \qquad n = 1, 2, ..., \\ b_{n+N}^0 &= b_n^0 \qquad n = 0, 1, 2, ... \end{aligned} \tag{2.1}$$

(here $N \geqslant 1$ is the period), and that the recurrence coefficients a_{n+1} and b_n $(n = 0, 1, ...)$ satisfy

$$\lim_{n\to\infty} |a_n - a_n^0| = 0, \qquad \lim_{n\to\infty} |b_n - b_n^0| = 0. \tag{2.2}$$

We say that the orthogonal polynomials have asymptotically periodic recurrence coefficients. In the literature (see, e.g., Geronimus [9]) the terminology *limit periodic* is also in use, but this is confusing because a limit periodic sequence (in the theory of almost periodic functions) is the uniform limit of periodic sequences, which is not what we mean by (2.2). We denote the orthonormal polynomials with periodic recurrence coefficients a_{n+1}^0, b_n^0 by $q_n(x)$. Define

$$\omega^N = \omega^N(x) = \rho\left(\frac{1}{2}\left\{q_N(x) - \frac{a_N^0}{a_{N+1}^0}\, q_{N-2}^{(1)}(x)\right\}\right), \tag{2.3}$$

where

$$\rho(x) = x + \sqrt{x^2 - 1}, \tag{2.4}$$

with the square root such that $\rho(x)$ is analytic with $|\rho(x)| > 1$ in $\mathbf{C} \setminus [-1, 1]$. On $[-1, 1]$ we define $\rho(x) = \rho(x + i0+)$, which gives $\rho(\cos\theta) = e^{i\theta}$. In particular for $x \in [-1, 1]$ one has $\Re[\rho(x)] = x$ and $\Im[\rho(x)] = \sqrt{1 - x^2}$, this square root being positive for every $x \in [-1, 1]$. Let

$$T(x) = \frac{1}{2}\left\{q_N(x) - \frac{a_N^0}{a_{N+1}^0}\, q_{N-2}^{(1)}(x)\right\},$$

and define E as the set where $|\omega(x)| = 1$; then E consists of the N intervals where

$$-1 \leqslant T(x) \leqslant 1,$$

and between every two consecutive intervals of E there is exactly one zero of each of the polynomials $T'(x)$ and $q_{N-1}^{(j)}(x)$ $(j \geqslant 0)$ (see [7, Lemma 2, pp. 255, 256]). On E we have $\Re[\omega^N(x)] = T(x)$ and $\Im[\omega^N(x)] = \operatorname{sign}[T'(x)]\sqrt{1 - T^2(x)}$, where the latter square root is always positive on E. The set E corresponds to the essential spectrum of the orthogonal polynomials $p_n(x)$ and the measure μ for these orthogonal polynomials has support $E \cup E^*$, where E^* is a denumerable set for which the accumulation points are on E. The set $\{\omega^{2N}(x) = 1\}$ consists of the endpoints of the intervals (it is possible that some of the intervals are touching at a point where $\omega^{2N}(x) = 1$). The orthogonality of the polynomials $q_n(x)$ is given by

$$\frac{1}{a_N^0 \pi} \int_E q_n(x) q_m(x) \frac{\sqrt{1 - T^2(x)}}{|q_{N-1}(x)|}\, dx + \frac{2}{a_N^0} \sum q_n(x_i) q_m(x_i) \frac{\sqrt{T^2(x_i) - 1}}{q'_{N-1}(x_i)} = \delta_{m,n}, \tag{2.5}$$

where x_i are those zeros of $q_{N-1}(x)$ for which $q_N(x_i) \neq \omega^N(x_i)$ [7, Theorem 5, p. 273; 10, Theorem 2]. This means that $E_0^* = \{x_i\}$ has at most $N-1$ points. The functions

$$\begin{aligned} q_n^+(x) &= q_{n+N}(x) - \omega^N(x) q_n(x) \\ q_n^-(x) &= q_{n+N}(x) - \omega^{-N}(x) q_n(x) \end{aligned} \tag{2.6}$$

are linearly independent solutions of the recurrence relation with periodic recurrence coefficients a_{n+1}^0, b_n^0 whenever x is not a zero of $q_{N-1}(x)$ and $\omega^{2N}(x) \neq 1$, and they have the property that $q_{nN+k}^+(x) = \omega^{-nN}(x) q_k^+(x)$ and $q_{nN+k}^-(x) = \omega^{nN}(x) q_k^-(x)$ [7, Theorem 1, p. 259]. This means that any solution of the recurrence relation with periodic recurrence coefficients can be written as a linear combination of $q_n^+(x)$ and $q_n^-(x)$, except possibly for finitely many values of x. Note that [7, Eq. (II.31), p. 261]

$$|q_n^+(x)|^2 = \frac{a_N^0}{a_{n+1}^0} q_{N-1}(x) q_{N-1}^{(n+1)}(x). \tag{2.7}$$

The following result generalizes a known result by Nevai [20, Theorem 13, p. 45] (see also Máté, Nevai, and Totik [16, Theorem 11.1, p. 270]) about Toeplitz matrices associated with orthogonal polynomials with converging recurrence coefficients:

LEMMA 1. *Suppose that* (2.2) *holds; then for every continuous function f and for all integers k and j one has*

$$\begin{aligned} \lim_{n\to\infty} \int f(x) p_{nN+j}(x) p_{nN+k}(x)\, d\mu(x) &= \frac{1}{4\pi a_{j+1}^0 a_{k+1}^0} \int_E f(x) \operatorname{sign}[T'(x)] \\ &\quad\times \frac{a_{k+1}^0 q_{k-j+N-1}^{(j+1)}(x) + a_{j+1}^0 q_{j-k+N-1}^{(k+1)}(x)}{\sqrt{1-T^2(x)}}\, dx, \end{aligned} \tag{2.8}$$

where $a_{m+k+1}^0 q_k^{(m)}(x) = -a_m^0 q_{-k-2}^{(m+k+1)}(x)$ *whenever* $k < 0$.

Proof. Denote by J the infinite Jacobi matrix

$$J = \begin{pmatrix} b_0 & a_1 & & & \\ a_1 & b_1 & a_2 & & \\ & a_2 & b_2 & a_3 & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

and by J_0 the infinite Jacobi matrix with the periodic recurrence coefficients a_{n+1}^0, b_n^0; then $J: l_2 \to l_2$ and $J_0: l_2 \to l_2$ are both bounded and selfadjoint linear operators on the Hilbert space l_2 of square summable sequences $(y_n: n = 0, 1, 2, ...)$. The condition (2.2) means that $J - J_0$ is a *compact* *operator* [12, p. 157]. Let $E \cup E^*$ be the spectrum of J and $E \cup E_0^*$ be the spectrum of J_0; then $(z-J)^{-1}$ is a bounded linear operator for $z \notin E \cup E^*$ and $(z-J_0)^{-1}$ is a bounded linear operator for $z \notin E \cup E_0^*$. Clearly $(z-J)^{-1}-(z-J_0)^{-1}=(z-J)^{-1}(J-J_0)(z-J_0)^{-1}$, and thus by [12, Theorem 4.8, p. 158] it follows that $(z-J)^{-1}-(z-J_0)^{-1}$ is a compact operator for $z \notin E \cup E^* \cup E_0^*$. If $A: l_2 \to l_2$ is compact, then one has $A_{n,n+k} \to 0$ as $n \to \infty$ for every integer k. Moreoer,

$$(z-J)^{-1}_{n,n+k}=\int \frac{p_n(x)\,p_{n+k}(x)}{z-x}\,d\mu(x),$$

$$(z-J_0)^{-1}_{n,n+k}=\int \frac{q_n(x)q_{n+k}(x)}{z-x}\,d\mu_0(x),$$

where μ_0 is given by (2.5). Therefore the result for $f(x)=(z-x)^{-1}$ follows for all orthogonal polynomials $p_n(x)$ satisfying (2.2) if and only if it is true for the polynomials $q_n(x)$ with periodic recurrence coefficients. Assume that $k \geqslant 0$; then by orthogonality

$$\int \frac{q_n(x)q_{n+k}(x)}{z-x}\,d\mu_0(x)=q_n(z)\int \frac{q_{n+k}(x)}{z-x}\,d\mu_0(x).$$

The functions

$$Q_n(z)=\int \frac{q_n(x)}{z-x}\,d\mu_0(x)$$

are known as *functions of the second kind* corresponding with the orthogonal polynomials $q_n(x)$ or the measure μ_0. For $z \notin E \cup E_0^*$ they are a *minimal solution* of the recurrence formula with periodic recurrence coefficients with initial conditions $Q_0(z)=S(z;\mu_0)$ (the Stieltjes transform of μ_0) and $a_0^0 Q_{-1}(z)=1$. This means that $Q_n(z)$ is a linear combination of $q_n^+(z)$ and $q_n^-(z)$, and because of the minimality and the initial conditions we have

$$a_N^0 q_{N-1}(x)Q_n(z)=q_n^+(z).$$

This together with the asymptotic formula [7, Theorem 8, p. 280]

$$\lim_{n\to\infty} q_{nN+j}(x)\omega^{-nN}(x)=\frac{a_N^0 q_{N-1}^{(j+1)}(x)q_{N-1}(x)}{a_{j+1}^0 q_j^+(z)(\omega^N(x)-\omega^{-N}(x))}$$

gives for $z \notin E \cup E_0^*$ and $k \geqslant j$

$$\lim_{n\to\infty}\int \frac{q_{nN+j}(x)q_{nN+k}(x)}{z-x}\,d\mu_0(x)=\frac{1}{2\sqrt{T^2(z)-1}}\,\frac{q_{N-1}^{(j+1)}(z)q_k^+(z)}{a_{j+1}^0 q_j^+(z)}.$$

By the Stieltjes inversion formula (e.g., [23, p. 175]) we have

$$\frac{1}{2\sqrt{T^2(z)-1}}\,\frac{q_{N-1}^{(j+1)}(z)q_k^+(z)}{a_{j+1}^0 q_j^+(z)} = \frac{1}{2a_{j+1}^0\pi}\int_E \Re\left(\frac{q_k^+(x)}{q_j^+(x)}\right)\frac{q_{N-1}^{(j+1)}(x)}{\sqrt{1-T^2(x)}}\,\mathrm{sign}[T'(x)]\,\frac{dx}{z-x},$$

and on E we have

$$\Re\left(\frac{q_k^+(x)}{q_j^+(x)}\right) = |q_j^+(x)|^{-2}\,\Re[q_k^+(x)q_j^-(x)]$$
$$= |q_j^+(x)|^{-2}\,[q_{k+N}(x)q_{j+N}(x)+q_k(x)q_j(x) - T(x)q_k(x)q_{j+N}(x) - T(x)q_{k+N}(x)q_j(x)].$$

If we use the relation $q_{n+2N}(x) = 2T(x)q_{n+N}(x) - q_n(x)$ [7, Lemma 1, p. 253] then

$$\Re[q_k^+(x)q_j^-(x)] = \tfrac{1}{2}[q_{k+N}(x)q_{j+N}(x) - q_k(x)q_{j+2N}(x)] + \tfrac{1}{2}[q_{k+N}(x)q_{j+N}(x) - q_j(x)q_{k+2N}(x)].$$

One can show [7, p. 260] that

$$\frac{a_N^0}{a_{m+1}^0}\,q_{N-1}(x)q_{n-m-1}^{(m+1)}(x) = [q_{m+N}(x)q_n(x) - q_m(x)q_{n+N}(x)].$$

If we combine the last four displays and if we use (2.7), then this all leads to

$$\frac{1}{2\sqrt{T^2(z)-1}}\,\frac{q_{N-1}^{(j+1)}(z)q_k^+(z)}{a_{j+1}^0 q_j^+(z)} = \frac{1}{4\pi a_{j+1}^0 a_{k+1}^0}\int_E \frac{a_{k+1}^0 q_{k-j+N-1}^{(j+1)}(x) + a_{j+1}^0 q_{j-k+N-1}^{(k+1)}(x)}{\sqrt{1-T^2(x)}} \times \mathrm{sign}[T'(x)]\,\frac{dx}{z-x}. \tag{2.9}$$

This proves the result when $f(x) = (z-x)^{-1}$, with $z \notin E \cup E^*$. From Schwarz' inequality and from the orthonormality it follows that the signed measures μ_n given by

$$\int f(x)\,d\mu_n(x) = \int f(x)\,p_{nN+k}(x)\,p_{nN+j}(x)\,d\mu(x)$$

with f a continuous function, have a total variation which is bounded by 1. Therefore, as a standard result for weak convergence of measures, the result holds for every continuous function f. ∎

Our aim is to generalize Theorem 2 for the case under consideration. Define

$$\psi_n(x; N) = p_{n+N}(x) - \left(\prod_{i=n+1}^{n+N} \frac{a_i^0}{a_i} \right) \omega^N(x)\, p_n(x), \tag{2.10}$$

where $\omega(x)$ is given by (2.3). For the special case where $T(x) = T_N(x)$—the Chebyshev polynomial of the first kind of degree N—this function $\psi_n(x; N)$ is essentially equal to the *twisted difference* $\Delta_{N,t}\, p_n$ introduced by Máté, Nevai, and Totik [17, p. 324]; the only difference is the factor in front of $\omega^N(x)\, p_n(x)$. The following result holds:

THEOREM 3. *Suppose that* (2.2) *holds and that*

$$\sum_{k=0}^{\infty} \left\{ |b_k - b_k^0| + \left| 1 - \left(\frac{a_{k+1}}{a_{k+1}^0} \right)^2 \right| \right\} < \infty. \tag{2.11}$$

Then μ is absolutely continuous on E, $w(x) = \mu'(x)$ is strictly positive and continuous on $E \setminus \{\omega^{2N}(x) = 1\}$, and

$$\lim_{n \to \infty} |\psi_{nN+j}(x; N)|^2 = \frac{1}{\pi} \frac{|q_{N-1}^{(j+1)}(x)|}{a_{j+1}^0 w(x)} \sqrt{1 - T^2(x)} \tag{2.12}$$

holds uniformly on every closed subset of $E \setminus \{\omega^{2N}(x) = 1\}$. If

$$\sum_{k=0}^{\infty} (k+1) \left\{ |b_k - b_k^0| + \left| 1 - \left(\frac{a_{k+1}}{a_{k+1}^0} \right)^2 \right| \right\} < \infty, \tag{2.13}$$

then (2.12) *holds uniformly on E. An upper bound for the error of approximation for $n \equiv j$* (mod N) *is given by*

$$\left| |\psi_n(x; N)| - \left(\prod_{i=n+N+1}^{\infty} \frac{a_i}{a_i^0} \right) \sqrt{\frac{1}{\pi} \frac{|q_{N-1}^{(j+1)}(x)|}{a_{j+1}^0 w(x)} \sqrt{1 - T^2(x)}} \right|$$
$$\leqslant C \left(\sum_{k=n}^{\infty} c_k \frac{k+N}{N + |1 - \omega^{-2N}(x)|\,(k+N)} \right)$$
$$\times \exp\left\{ A \sum_{k=0}^{\infty} c_k \frac{k+N}{N + |1 - \omega^{-2N}(x)|\,(k+N)} \right\}, \tag{2.14}$$

where

$$c_k = \left| \frac{b_k - b_k^0}{a_{k+1}^0} \right| + \frac{a_{k+1}^0}{a_{k+2}^0} \left| 1 - \left(\frac{a_{k+1}}{a_{k+1}^0} \right)^2 \right|.$$

Proof. The absolute continuity of μ and the positivity of $w(x)$ on E are shown in [7, Theorem 6, p. 277]. Introduce

$$\hat{p}_n(x) = \left(\prod_{i=1}^{n} \frac{a_i}{a_i^0} \right) p_n(x);$$

then, by the method of variation of constants, one has the comparison equation [7, Eq. (III.8), p. 263]

$$\hat{p}_n(x) = q_n(x) + \sum_{k=0}^{n-1} \left\{ \frac{b_k^0 - b_k}{a_{k+1}^0} q_{n-k-1}^{(k+1)}(x) + \frac{a_{k+1}^0}{a_{k+2}^0} \left\{ 1 - \left(\frac{a_{k+1}}{a_{k+1}^0} \right)^2 \right\} q_{n-k-2}^{(k+2)}(x) \right\} \hat{p}_k(x). \qquad (2.15)$$

Replace n by $n+N$ in this equation and subtract (2.15) multiplied by $\omega^N(x)$ to find

$$\begin{aligned} \hat{p}_{n+N}(x) - \omega^N(x)\,\hat{p}_n(x) &= q_{n+N}(x) - \omega^N(x) q_n(x) \\ &+ \sum_{k=0}^{n-1} \left\{ \frac{b_k^0 - b_k}{a_{k+1}^0} [q_{n+N-k-1}^{(k+1)}(x) - \omega^N(x) q_{n+k-1}^{(k+1)}(x)] \right. \\ &\left. + \frac{a_{k+1}^0}{a_{k+2}^0} \left\{ 1 - \left(\frac{a_{k+1}}{a_{k+1}^0} \right)^2 \right\} [q_{n+N-k-2}^{(k+2)}(x) - \omega^N(x) q_{n+k-2}^{(k+2)}(x)] \right\} \hat{p}_k(x) \\ &+ \sum_{k=n}^{n+N-1} \left\{ \frac{b_k^0 - b_k}{a_{k+1}^0} q_{n+N-k-1}^{(k+1)}(x) \right. \\ &\left. + \frac{a_{k+1}^0}{a_{k+2}^0} \left\{ 1 - \left(\frac{a_{k+1}}{a_{k+1}^0} \right)^2 \right\} q_{n+N-k-2}^{(k+2)}(x) \right\} \hat{p}_k(x). \end{aligned}$$

The sequence $q_{n+N-k-1}^{(k+1)}(x) - \omega^N(x) q_{n-k-1}^{(k+1)}(x)$ $(n = k,\ k+1, ...)$ is also a solution of the periodic recurrence formula and has the same periodic behaviour as $q_n^+(x)$. Therefore this sequence is equal to $q_n^+(x)$ up to a multiplicative factor (that may depend on k and x, but is independent of n). One has explicitly

$$q_{n+N-k-1}^{(k+1)}(x) - \omega^N(x) q_{n-k-1}^{(k+1)}(x) = \frac{q_{N-1}^{(k+1)}(x)}{q_k^+(x)} q_n^+(x).$$

Recall that q_k^+ has no zeros in the interior of E (use (2.7) and the fact that $q_{N-1}^{(j)}$ has no zeros in the interior of E). If we use this formula, then

$$\begin{aligned}\hat{p}_{n+N}(x)-\omega^N(x)\,\hat{p}_n(x)=q_n^+(x)\Bigg(1+\sum_{k=0}^{n-1}\Bigg\{&\frac{b_k^0-b_k}{a_{k+1}^0}\frac{q_{N-1}^{(k+1)}(x)}{q_k^+(x)}\\&+\frac{a_{k+1}^0}{a_{k+2}^0}\left\{1-\left(\frac{a_{k+1}}{a_{k+1}^0}\right)^2\right\}\frac{q_{N-1}^{(k+2)}(x)}{q_{k+1}^+(x)}\Bigg\}\,\hat{p}_k(x)\\&+\sum_{k=n}^{n+N-1}\Bigg\{\frac{b_k^0-b_k}{a_{k+1}^0}\,q_{n+N-k-1}^{(k+1)}(x)\\&+\frac{a_{k+1}^0}{a_{k+2}^0}\left\{1-\left(\frac{a_{k+1}}{a_{k+1}^0}\right)^2\right\}q_{n+N-k-2}^{(k+2)}(x)\Bigg\}\,\hat{p}_k(x)\Bigg).\end{aligned} \tag{2.16}$$

On E we have the bound [7, Eq. (III.10), p. 263]

$$\begin{aligned}|\hat{p}_n(x)|\leqslant&\frac{A(n+N)}{N+|1-\omega^{-2N}(x)|\,(n+N)}\\&\times\exp\left\{A\sum_{k=0}^{n-1}c_k\frac{(k+N)}{N+|1-\omega^{-2N}(x)|\,(k+N)}\right\}.\end{aligned} \tag{2.17}$$

This bound, the property (2.7), and the periodicity of the coefficients a_{n+1}^0, b_n^0 show that the series

$$\begin{aligned}\psi(x)=1+\sum_{k=0}^{\infty}\Bigg\{&\frac{b_k^0-b_k}{a_{k+1}^0}\frac{q_{N-1}^{(k+1)}(x)}{q_k^+(x)}\\&+\frac{a_{k+1}^0}{a_{k+2}^0}\left\{1-\left(\frac{a_{k+1}}{a_{k+1}^0}\right)^2\right\}\frac{q_{N-1}^{(k+2)}(x)}{q_{k+1}^+(x)}\Bigg\}\,\hat{p}_k(x)\end{aligned} \tag{2.18}$$

converges uniformly on closed subsets of $E\backslash\{\omega^{2N}(x)=1\}$ whenever (2.11) holds, and uniformly on E when the stronger condition (2.13) is true. The bounds (2.17) and

$$|q_n^{(k)}(x)|\leqslant\frac{A(n+N)}{N+|1-\omega^{-2N}(x)|\,(n+N)}\qquad(x\in E)$$

[7, Eq. (II.21), p. 258], together with (2.16), then give

$$\begin{aligned}&|[\hat{p}_{n+N}(x)-\omega^N(x)\,\hat{p}_n(x)]-q_n^+(x)\psi(x)|\\&\quad\leqslant\sum_{k=n}^{\infty}c_k\frac{k+N}{N+|1-\omega^{-2N}(x)|\,(k+N)}\\&\qquad\times C\exp\left\{A\sum_{k=0}^{\infty}c_k\frac{k+N}{N+|1-\omega^{-2N}(x)|\,(k+N)}\right\}.\end{aligned} \tag{2.19}$$

This and (2.7) show that

$$\lim_{n\to\infty} |\hat{p}_{(n+1)N+k}(x) - \omega^N(x)\,\hat{p}_{nN+k}(x)|^2 = \frac{a_N^0}{a_{k+1}^0}\, q_{N-1}(x) q_{N-1}^{(k+1)}(x)\, |\psi(x)|^2 \tag{2.20}$$

holds uniformly on closed subsets of $E\backslash\{\omega^{2N}(x)=1\}$ when (2.11) holds and uniformly on E when (2.13) is true. Finally, (2.12) then follows from Lemma 1 and moreover

$$|\psi(x)|^2 = \frac{1}{\pi}\frac{\gamma^2}{a_N^0 w(x)} \frac{\sqrt{1-T^2(x)}}{|q_{N-1}(x)|},$$

where

$$\gamma = \prod_{i=1}^{\infty} \frac{a_i}{a_i^0}.$$

Indeed, from Lemma 1 we have for $f \in C(E)$ (take $f(x)=0$ when $x \notin E$)

$$\lim_{n\to\infty} \int_E f(x)\, |\psi_{nN+j}(x; N)|^2\, w(x)\, dx$$
$$= \frac{1}{\pi}\frac{1}{a_{j+1}^0} \int_E f(x) \frac{q_{N-1}^{(j+1)}(x) - \frac{1}{2}T(x) q_{2N-1}^{(j+1)}(x)}{\sqrt{1-T^2(x)}} \operatorname{sign}[T'(x)]\, dx,$$

and by using $q_{n+2N}^{(j+1)}(x) = 2T(x) q_{n+N}^{(j+1)}(x) - q_n^{(j+1)}(x)$ [7, Lemma 1, p. 253] this gives

$$\lim_{n\to\infty} \int_E f(x)\, |\psi_{nN+j}(x; N)|^2\, w(x)\, dx$$
$$= \frac{1}{\pi}\frac{1}{a_{j+1}^0} \int_E f(x)\, |q_{N-1}^{(j+1)}(x)|\, \sqrt{1-T^2(x)}\, dx.$$

The relation (2.12) then holds because the last asymptotic formula is valid for every continuous function f on E. The upper bound (2.14) can be obtained from (2.19). ∎

3. Recurrence Coefficients of Bounded Variation (modulo N)

Suppose that the recurrence coefficients are asymptotically periodic (period N), so that (2.2) holds. In addition to this, we also assume that the coefficients are of bounded variation (modulo N), meaning

$$\sum_{k=0}^{\infty} (|b_k - b_{k+N}| + |a_{k+1} - a_{k+N+1}|) < \infty. \tag{3.1}$$

Various results for such orthogonal polynomials have been proved for the case when $N=1$ (Máté and Nevai [14], Máté, Nevai, and Totik [15], Dombrowski and Nevai [4], Van Assche and Geronimo [26]). Our aim is to extend these results to the case when $N>1$. The idea here is to use not the second order recurrence formula (1.2) but a higher order recurrence formula. A useful formula is

$$p_{n+2N}(x)\, p_{N-1}^{(n+1)}(x) = p_{n+N}(x)\, p_{2N-1}^{(n+1)}(x) - \frac{a_{n+1}}{a_{n+N+1}}\, p_n(x)\, p_{N-1}^{(n+N+1)}(x) \tag{3.2}$$

(see, e.g., [7, Lemma 11, p. 278]) but the polynomial $p_{N-1}^{(n+1)}(x)$ multiplying $p_{n+2N}(x)$ is somewhat annoying. A better formula is given by:

LEMMA 2. *Suppose that $p_n(x)$ $(n=0, 1, 2, ...)$ are orthogonal polynomials satisfying the recurrence formula* (1.2); *then*

$$p_{n+2N}(x) = A_n(x; N)\, p_{n+N}(x) - B_n(x; N)\, p_n(x) - C_n(x; N)\, p_{n-1}(x), \tag{3.3}$$

where

$$A_n(x; N) = p_N^{(n+N)}(x) - \frac{a_{n+2N}}{a_{n+N+1}}\, p_{N-2}^{(n+N+1)}(x),$$

$$B_n(x; N) = \frac{1}{a_{n+N+1}} \left(a_{n+N}\, p_{N-1}^{(n+N+1)}(x)\, p_{N-1}^{(n)}(x) - a_{n+2N}\, p_{N-2}^{(n+N+1)}(x)\, p_N^{(n)}(x)\right),$$

$$C_n(x; N) = \frac{1}{a_{n+N+1}} \frac{a_n}{a_{n+1}} \Big(a_{n+2N}\, p_{N-2}^{(n+N+1)}(x)\, p_{N-1}^{(n+1)}(x) - a_{n+N}\, p_{N-1}^{(n+N+1)}(x)\, p_{N-2}^{(n+1)}(x)\Big).$$

Proof. We start with the equation [7, Eq. (V.2), p. 278]

$$p_k(x) = p_m(x)\, p_{k-m}^{(m)}(x) - \frac{a_m}{a_{m+1}}\, p_{m-1}(x)\, p_{k-m-1}^{(m+1)}(x). \tag{3.4}$$

Set $k=n+2N$ and $m=n+N$, then add and substract (a_{n+2N}/a_{n+N+1}) $p_{N-2}^{(n+N+1)}(x)\, p_{n+N}(x)$ to find

$$p_{n+2N}(x) = A_n(x; N)\, p_{n+N}(x) - \frac{a_{n+N}}{a_{n+N+1}}\, p_{n+N-1}(x)\, p_{N-1}^{(n+N+1)}(x) + \frac{a_{n+2N}}{a_{n+N-1}}\, p_{n+N}(x)\, p_{N-2}^{(n+N+1)}(x).$$

Now use (3.4) with $k=n+N$, $m=n$ and $k=n+N-1$, $m=n$ to replace $p_{n+N}(x)$ and $p_{n+N-1}(x)$ by $p_n(x)$ and $p_{n-1}(x)$; then the lemma follows. ∎

It is clear that the asymptotic periodicity (2.2) implies that

$$\lim_{\substack{n\to\infty \\ n\equiv j \pmod N}} A_n(x;N)=q_N^{(j)}(x)-\frac{a_j^0}{a_{j+1}^0}\,q_{N-2}^{(j+1)}(x)=2T(x),$$

$$\lim_{\substack{n\to\infty \\ n\equiv j \pmod N}} B_n(x;N)=\frac{a_j^0}{a_{j+1}^0}\,(q_{N-1}^{(j+1)}(x)q_{N-1}^{(j)}(x)-q_{N-2}^{(j+1)}(x)q_N^{(j)}(x))=1,$$

$$\lim_{\substack{n\to\infty \\ n\equiv j \pmod N}} C_n(x;N)=0.$$

For the first of these limit relations we have used the fact that

$$2T(x)=q_N(x)-\frac{a_N^0}{a_{N+1}^0}\,q_{N-2}^{(1)}(x)=q_N^{(j)}(x)-\frac{a_{N+j}^0}{a_{N+j+1}^0}\,q_{N-2}^{(j+1)}(x) \tag{3.5}$$

[7, Corollary 1, p. 254], and for the limit of $B_n(x;N)$ we have used the Wronskian of $q_{n-j}^{(j)}(x)$ and $q_{n-j-1}^{(j+1)}(x)$. We need good bounds on the convergence of some of these expressions. These bounds are given by:

LEMMA 3. *Suppose K is a compact set in the complex plane; then there exists a constant C such that for $x\in K$*

$$|C_n(x)|\leqslant C\sum_{k=n+1}^{n+N-1}(|b_k-b_{k+N}|+|a_{k+1}-a_{k+N+1}|), \tag{3.6}$$

$$|1-B_n(x;N)|\leqslant C\sum_{k=n}^{n+N-1}(|b_k-b_{k+N}|+|a_{k+1}-a_{k+N+1}|). \tag{3.7}$$

Proof. Consider the function

$$R_i(x)=p_i^{(n+1)}(x)-p_i^{(n+N+1)}(x), \tag{3.8}$$

then for $x\in K$

$$R_0(x)=0,$$

$$|R_1(x)|=\left|\frac{x-b_{n+1}}{a_{n+2}}-\frac{x-b_{n+N+1}}{a_{n+N+2}}\right|$$
$$\leqslant C_1(|b_{n+1}-b_{n+N+1}|+|a_{n+2}-a_{n+N+2}|),$$

where C_1 is a constant (depending on the set K). By using the recurrence formula (1.2) one also finds

$$R_i(x)=\left(\frac{x-b_{n+i}}{a_{n+i+1}}-\frac{x-b_{n+N+i}}{a_{n+N+i+1}}\right)p_{i-1}^{(n+1)}(x)+\frac{x-b_{n+N+1}}{a_{n+N+i+1}}R_{i-1}(x)$$
$$-\left(\frac{a_{n+i}}{a_{n+i+1}}-\frac{a_{n+N+i}}{a_{n+N+i+1}}\right)p_{i-2}^{(n+1)}(x)+\frac{a_{n+N+i}}{a_{n+N+i+1}}R_{i-2}(x).$$

From this one can easily show by induction that

$$|R_i(x)|\leqslant C_i\sum_{k=n+1}^{n+i}(|b_k-b_{k+N}|+|a_{k+1}-a_{k+N+1}|),\qquad x\in K,\qquad(3.9)$$

where C_i is independent of n and x. From this the bound (3.6) follows immediately. For (3.7) we note that

$$B_n(x;N)=\frac{a_{n+N}}{a_{n+N+1}}(p_{N-1}^{(n+1)}(x)\,p_{N-1}^{(n)}(x)-p_{N-2}^{(n+1)}(x)\,p_N^{(n)}(x))$$
$$+\frac{a_{n+N}}{a_{n+N+1}}[R_{N-2}(x)-R_{N-1}(x)]$$
$$+\frac{p_{N-2}^{(n+N-1)}(x)}{a_{n+N+1}}(a_{n+N}-a_{n+2N}).$$

The first term on the right hand side is just the Wronskian of two solutions of the recurrence formula (1.2), and hence

$$B_n(x;N)=\frac{a_{n+1}}{a_{n+N+1}}+\frac{a_{n+N}}{a_{n+N+1}}[R_{N-2}(x)-R_{N-1}(x)]$$
$$+\frac{p_{N-2}^{(n+N-1)}(x)}{a_{n+N+1}}(a_{n+N}-a_{n+2N}).$$

The bound (3.7) now follows immediately from the bounds (3.9). ∎

Following ideas in [25, Sect. 3, p. 450] and [26, Sect. III, p. 226] we introduce the functions

$$G_j(k,m)=\sum_{i=k+1}^{m}\left(\prod_{l=i+1}^{m}\frac{1}{t_{lN+j}}\right)\left(\prod_{l=k+1}^{i-1}t_{lN+j}\right)\qquad j=0,1,...,N-1,\qquad(3.10)$$

where

$$t_n=\rho\left(\frac{1}{2}\left(p_N^{(n)}(x)-\frac{a_{n+N}}{a_{n+1}}p_{N-2}^{(n+1)}(x)\right)\right)\qquad(3.11)$$

and $\rho(x)$ is given by (2.4). We want to compare the orthogonal polynomials $\{p_n(x) : n=0, 1, 2, ...\}$ with these *Green's functions* $\{G_j(k, m) : k<m, j=0, 1, ..., N-1\}$. One of the reasons for doing so is that these Green's functions are explicitly defined in terms of the recurrence coefficents and that one can obtain nice bounds for them:

LEMMA 4. *Let K be a compact set either in $\mathbf{C}^+ \setminus \{\omega^{2N}(x)=1\}$ or in $\mathbf{C}^- \setminus \{\omega^{2N}(x)=1\}$, where $\mathbf{C}^{+(-)}=\{z \in \mathbf{C} : \Im z \geqslant (\leqslant) 0\}$; then there exist constants C and D (depending only on the set K and the period N) such that for $x \in K$*

$$\left|\frac{G_j(k, m)}{\prod_{i=k+1}^{m+1} t_{iN+j}}\right| \leqslant C \exp\left\{D \sum_{i=(k+1)N+j-1}^{(m+1)N+j} (|b_i - b_{i+N}| + |a_{i+1} - a_{i+N+1}|)\right\}. \tag{3.12}$$

Proof. From (3.10) one easily finds that for $k<m$

$$t_{mN+j} G_j(k, m) - G_j(k, m-1) = \prod_{l=k+1}^{m} t_{lN+j}. \tag{3.13}$$

If we define

$$\hat{G}_j(k, m) = \frac{G_j(k, m)}{\prod_{l=k+1}^{m+1} t_{lN+j}},$$

then this becomes

$$t_{(m+1)N+j} t_{mN+j} \hat{G}_j(k, m) - \hat{G}_j(k, m-1) = 1, \tag{3.14}$$

which means that the left hand side is independent of m. Equate the expressions for m and $m-1$; then one finds that

$$\hat{G}_j(k, m) - \hat{G}_j(k, m-1) = \frac{\hat{G}_j(k, m-1) - \hat{G}_j(k, m-2)}{t_{(m+1)N+j} t_{mN+j}} + \frac{t_{(m-1)N+j} - t_{(m+1)N+j}}{t_{(m+1)N+j}} \hat{G}_j(k, m-1).$$

Since $|t_n| \geqslant 1$ for every n, one obtains

$$|\hat{G}_j(k, m) - \hat{G}_j(k, m-1)| \leqslant |\hat{G}_j(k, m-1) - \hat{G}_j(k, m-2)| + |t_{(m-1)N+j} - t_{(m+1)N+j}|\ |\hat{G}_j(k, m-1)|.$$

Iteration then gives

$$|\hat{G}_j(k, m) - \hat{G}_j(k, m-1)| \leqslant 1 + \sum_{i=k+1}^{m-1} |t_{(i+2)N+j} - t_{iN+j}|\ |\hat{G}_j(k, i)|,$$

where we have used $G_j(k,k)=0$ and $G_j(k,k+1)=1$. From (3.14) we have

$$\hat{G}_j(k,m)-\hat{G}_j(k,m-1)=1+\hat{G}_j(k,m)(1-t_{mN+j}t_{(m+1)N+j}),$$

from which one deduces

$$|\hat{G}_j(k,m)-\hat{G}_j(k,m-1)|\geqslant|\hat{G}_j(k,m)|\;|1-t_{mN+j}t_{(m+1)N+j}|-1,$$

and this inequality thus gives

$$|\hat{G}_j(k,m)|\;|1-t_{mN+j}t_{(m+1)N+j}|\leqslant 2+\sum_{i=k+1}^{m-1}|t_{(i+2)N+j}-t_{iN+j}|\;|\hat{G}_j(k,i)|.$$

Since $t_n\to\omega^N$ as $n\to\infty$—$\omega^N(x)$ is given by (2.3)—there is an integer $M>0$ and a positive constant C_1 such that $|1-t_{mN+j}t_{(m+1)N+j}|^{-1}\leqslant C_1$ for $m>M$ and $x\in K$, therefore

$$|\hat{G}_j(k,m)|\leqslant 2C_1+C_1\sum_{i=k+1}^{m-1}|t_{(i+2)N+j}-t_{iN+j}|\;|\hat{G}_j(k,i)|.$$

Now use Gronwall's inequality (see, e.g., [25, p. 440]) to find

$$|\hat{G}_j(k,m)|\leqslant 2C_1\exp\left\{C_1\sum_{i=k+1}^{m-1}|t_{(i+2)N+j}-t_{iN+j}|\right\},$$

which holds for $m>k>M$. It is an easy exercise to show that $|\rho(x)-\rho(y)|\leqslant C_2\,|x-y|$ for some constant C_2 and for $x,y\in K$. The bound (3.9) and the definition (3.11) then give

$$|t_n-t_{n+N}|\leqslant C_3\sum_{l=n}^{n+N-1}(|b_l-b_{l+N}|+|a_{l+1}-a_{l+N+1}|),$$

from which the desired inequality follows. ∎

We have already mentioned the fact that

$$\lim_{n\to\infty}t_n=\rho(T(x))=\omega^N(x)$$

uniformly on compact sets of $\mathbf{C}\setminus\{\omega^{2N}(x)=1\}$, where $\omega^N(x)$ is given by (2.3). Note that there is no need here to let n tend to infinity along subsequences $n\equiv j\pmod N$. For the rate of convergence, we have

LEMMA 5. *Suppose K is a compact set either in $\mathbf{C}^+\setminus\{\omega^{2N}(x)=1\}$ or in $\mathbf{C}^-\setminus\{\omega^{2N}(x)=1\}$, where $\mathbf{C}^{+(-)}=\{z\in\mathbf{C}:\Im z\geqslant(\leqslant)\,0\}$; then there exists a constant C such that for $x\in K$*

$$|t_n-t_{n+1}|\leqslant C(|b_n-b_{n+N}|+|a_{n+1}-a_{n+N+1}|). \tag{3.15}$$

Proof. In order to prove (3.15) we first estimate the difference

$$E_n(x)=\left(p_N^{(n)}(x)-\frac{a_{n+N}}{a_{n+1}}\,p_{N-2}^{(n+1)}(x)\right)-\left(p_N^{(n+1)}(x)-\frac{a_{n+N+1}}{a_{n+2}}\,p_{N-2}^{(n+2)}(x)\right). \tag{3.16}$$

The equation (3.4) with $k=N$, $m=1$ but for the nth associated polynomials becomes

$$p_N^{(n)}(x)=p_1^{(n)}(x)\,p_{N-1}^{(n+1)}(x)-\frac{a_{n+1}}{a_{n+2}}\,p_{N-2}^{(n+2)}(x).$$

Use this to replace $p_N^{(n)}(x)$ in (3.16) and use the recurrence formula to replace $p_N^{(n+1)}(x)$; then

$$E_n(x)=\left(\frac{x-b_n}{a_{n+2}}-\frac{x-b_{n+N}}{a_{n+N+1}}\right)p_{N-1}^{(n+1)}(x)+\frac{a_{n+N+1}-a_{n+1}}{a_{n+2}}\,p_{N-2}^{(n+2)}(x)$$
$$+\left(\frac{a_{n+N}}{a_{n+N+1}}-\frac{a_{n+N}}{a_{n+1}}\right)p_{N-2}^{(n+1)}(x).$$

The result now follows because $p_{N-1}^{(n+1)}(x)$ and $p_{N-2}^{(n+1)}(x)$ are uniformly bounded on K and because $|\rho(x)-\rho(y)|\leqslant C\,|x-y|$ on K. ∎

The main reason that the Green's functions (3.10) are so useful as a comparison system for the orthogonal polynomials is that they satisfy a recurrence relation which is very similar to (3.3). Indeed, a companion to formula (3.13) is given by

$$G_j(k+1,m)-\frac{1}{t_{(k+1)N+j}}\,G_j(k,m)=-\prod_{l=k+1}^{m}\frac{1}{t_{lN+j}},\qquad k<m, \tag{3.17}$$

from which one easily obtains the three-term recurrence formula

$$G_j(k+1,m)+G_j(k-1,m)$$
$$=\left(t_{kN+j}+\frac{1}{t_{(k+1)N+j}}\right)G_j(k,m)+\delta_{k,m},\qquad k\leqslant m. \tag{3.18}$$

The recurrence formula (3.3) for $n=(k-1)N+j$ can be restated as

$$p_{(k+1)N+j}(x)=\left(t_{kN+j}+\frac{1}{t_{kN+j}}\right)p_{kN+j}(x)-B_{(k-1)N+j}(x;N)\,p_{(k-1)N+j}(x)$$
$$-C_{(k-1)N+j}(x;N)\,p_{(k-1)N+j-1}(x). \tag{3.19}$$

The similarity between (3.18) an (3.19) and the bounds given in Lemma 3 can be used to obtain bounds and the asymptotic behaviour for the orthogonal polynomials:

THEOREM 4. *Suppose that* (2.2) *and* (3.1) *hold, and let K be a compact set either in $\mathbf{C}^+\setminus\{\omega^{2N}(x)=1\}$ or in $\mathbf{C}^-\setminus\{\omega^{2N}(x)=1\}$, where $\mathbf{C}^{+(-)}=\{z\in\mathbf{C}:\Im z\geqslant(\leqslant)\,0\}$; then there exist constants C and D such that for $x\in K$*

$$\left|\frac{p_{mN+j}(x)}{\prod_{l=1}^{m+1} t_{lN+j}}\right|\leqslant C\exp\left\{D\sum_{i=0}^{mN+j-1}\left(|b_i-b_{i+N}|+|a_{i+1}-a_{i+N+1}|\right)\right\}. \quad (3.20)$$

Moreover, there exist continuous functions $\varphi_j(x)$ on K such that

$$\lim_{n\to\infty}\frac{t_{nN+j}\,p_{nN+j}(x)-p_{(n-1)N+j}(x)}{\prod_{l=1}^{n+1} t_{lN+j}}=\varphi_j(x), \qquad j=0,\ldots,N-1, \quad (3.21)$$

uniformly on K.

Proof. Multiply (3.18) by $p_{kN+j}(x)$ and (3.19) by $G_j(k,m)$, then subtract the obtained equations to find

$$\begin{aligned}p_{kN+j}(x)\delta_{k,m}&=G_j(k-1,m)\,p_{kN+j}(x)-G_j(k,m)\,p_{(k+1)N+j}(x)\\&\quad+G_j(k+1,m)\,p_{kN+j}(x)\\&\quad-B_{(k-1)N+j}(x;N)G_j(k,m)\,p_{(k-1)N+j}(x)\\&\quad+\left(\frac{1}{t_{kN+j}}-\frac{1}{t_{(k+1)N+j}}\right)G_j(k,m)\,p_{kN+j}(x)\\&\quad-C_{(k-1)N+j}(x;N)G_j(k,m)\,p_{(k-1)N+j-1}(x).\end{aligned}$$

Summing from $k=1$ to $k=m$ gives

$$\begin{aligned}p_{mN+j}(x)&=p_{N+j}(x)G_j(0,m)-p_j(x)G_j(1,m)+\left(\frac{1}{t_{N+j}}-\frac{1}{t_j}\right)p_j(x)G_j(0,m)\\&\quad+\sum_{k=0}^{m-1}[1-B_{kN+j}(x;N)]G_j(k+1,m)\,p_{kN+j}(x)\\&\quad+\sum_{k=0}^{m-1}\left(\frac{1}{t_{kN+j}}-\frac{1}{t_{(k+1)N+j}}\right)G_j(k,m)\,p_{kN+j}(x)\\&\quad-\sum_{k=0}^{m-1}C_{kN+j}(x;N)G_j(k+1,m)\,p_{kN+j-1}(x).\end{aligned} \quad (3.22)$$

Define

$$\hat{p}_{kN+j}(x) = \frac{p_{kN+j}(x)}{\prod_{l=1}^{k+1} t_{lN+j}},$$

then by Lemma 2, Lemma 3, and $|t_n| \geqslant 1$, it follows that

$$\begin{aligned} |\hat{p}_{mN+j}(x)| \leqslant C \exp\left\{D \sum_{i=0}^{(m+1)N+j-1} (|b_i - b_{i+N}| + |a_{i+1} - a_{i+N+1}|)\right\} \\ \times \left\{1 + \sum_{k=0}^{m-1} \left(\sum_{i=kN+j}^{(k+1)N+j-1} (|b_i - b_{i+N}| + |a_{i+1} - a_{i+N+1}|)\right)\right. \\ \times \Bigg((1 + |t_{(k+1)N+j}|)\, |\hat{p}_{kN+j}(x)| \\ \left. + \left|\prod_{l=1}^{k+1} \frac{t_{lN+j-1}}{t_{lN+j}}\right| |\hat{p}_{kN+j-1}(x)|\Bigg)\right\}. \end{aligned} \tag{3.23}$$

The inequality

$$\prod_{l=1}^{k+1} \left|\frac{t_{lN+j-1}}{t_{lN+j}}\right| \leqslant \exp\left\{\sum_{l=1}^{k+1} |t_{lN+j-1} - t_{lN+j}|\right\},$$

Eq. (3.1), and Gronwall's inequality (see, e.g., [25, p. 440]) then lead to the desired inequality (3.20).

Next, multiply both sides of (3.22) by t_{mN+j} and subtract (3.22) with m replaced by $m-1$. The obtained equality contains expressions of the form $t_{mN+j} G_j(k, m) - G_j(k, m-1)$; hence by using (3.13) we find

$$\begin{aligned} &\frac{t_{mN+j} p_{mN+j}(x) - p_{(m-1)N+j}(x)}{\prod_{i=1}^{m} t_{iN+j}} \\ &= p_{N+j}(x) - \frac{p_j(x)}{t_{N+j}} + \left(\frac{1}{t_{N+j}} - \frac{1}{t_j}\right) p_j(x) \\ &+ \sum_{k=0}^{m-2} \left\{1 - B_{kN+j}(x; N)\right. \\ &\left. + \left(\frac{1}{t_{kN+j}} - \frac{1}{t_{(k+1)N+j}}\right) t_{(k+1)N+j}\right\} \frac{p_{kN+j}(x)}{\prod_{i=1}^{k+1} t_{iN+j}} \\ &- \sum_{k=0}^{m-2} C_{kN+j}(x; N)\, t_{(k+1)N+j} \left(\prod_{i=1}^{k+1} \frac{t_{iN+j-1}}{t_{iN+j}}\right) \frac{p_{kN+j-1}(x)}{\prod_{i=1}^{k+1} t_{iN+j-1}} \\ &+ \left(\frac{1}{t_{(m-1)N+j}} - \frac{1}{t_{mN+j}}\right) \frac{p_{(m-1)N+j}(x)}{\prod_{i=1}^{m} t_{iN+j}}. \end{aligned} \tag{3.24}$$

It is clear from (3.20) that the series

$$\varphi_j(x)=p_{N+1}(x)-\frac{p_j(x)}{t_{N+j}}+\left(\frac{1}{t_{N+j}}-\frac{1}{t_j}\right)p_j(x)$$
$$+\sum_{k=0}^{\infty}\left\{1-B_{kN+j}(x;N)\right.$$
$$\left.+\left(\frac{1}{t_{kN+j}}-\frac{1}{t_{(k+1)N+j}}\right)t_{(k+1)N+j}\right\}\frac{p_{kN+j}(x)}{\prod_{i=1}^{k+1}t_{iN+j}}$$
$$-\sum_{k=0}^{m-2}C_{kN+j}(x;N)t_{(k+1)N+j}\left(\prod_{i=1}^{k+1}\frac{t_{iN+j-1}}{t_{iN+j}}\right)\frac{p_{kN+j-1}(x)}{\prod_{i=1}^{k+1}t_{iN+j-1}} \tag{3.25}$$

converges uniformly on K whenever (2.2) and (3.1) hold, and that

$$\lim_{n\to\infty}\frac{t_{nN+j}p_{nN+j}(x)-p_{(n-1)N+j}(x)}{\prod_{i=1}^{n+1}t_{iN+j}}=\varphi_j(x),$$

which proves the theorem. ∎

Let S be an infinite subset of the positive integers and let $\Omega(S)$ be the closure of the set of zeros of $p_n(x)$ as n runs through S. Then (2.2) implies that

$$\lim_{n\to\infty, n\in S}\frac{p_n(x)}{p_{n+N}(x)}=\omega^{-N}(x)$$

uniformly on compact sets of $\mathbf{C}\backslash\Omega(S)$, where $\omega^N(x)$ is given by (2.3) (see, e.g., [23, Theorem 2.22, p. 75]). Hence (3.21) implies

$$\lim_{n\to\infty,\, n\in S}\frac{p_{nN+j}(x)}{\prod_{i=1}^{n+1}t_{iN+j}}=\frac{\varphi_j(x)}{\omega^N(x)-\omega^{-N}(x)} \tag{3.26}$$

uniformly for x on compact sets of $\mathbf{C}\backslash\Omega(S)$ whenever both (2.2) and (3.1) hold. Following Máté, Nevai, and Van Assche [19, Section 3] we denote by Ξ the set of real numbers x such that for every neighborhood U of x there is an integer k such that for every $n>k$ the polynomial $p_n(x)$ has a zero in U. Then

$$E\subseteq\Xi\subseteq\sigma(J)\subseteq\Omega(\mathbf{Z}^+),$$

where $\sigma(J)$ is the spectrum of the Jacobi operator J, i.e., $\sigma(J)=\operatorname{supp}(\mu)=$

$E \cup \{\text{mass points of } \mu\}$. Recall that the mass points of μ are denumerable, outside $E \backslash \{\omega^{2N}(x)=1\}$ and possibly accumulating at $\{\omega^{2N}(x)=1\}$. For each point $x \in \sigma(J) \backslash E$ one knows that $\{p_n(x) : n=0, 1, 2, ...\} \in l_2$. Poincaré's theorem applied to (3.2) then implies that for such x

$$\lim_{n \to \infty} \frac{p_n(x)}{p_{n+N}(x)} = \omega^N(x),$$

where $\omega^N(x)$ is given by (2.3). The asymptotic behavior of $p_n(x)$ for $x \in \sigma(J) \backslash E$ cannot be obtained from (3.21) because as n tends to infinity, both sides of (3.21) tend to zero. If K is a compact set in $E \backslash \{\omega^{2N}(x)=1\}$, then there exists an integer n_j such that $|t_{nN+j}|=1$ for $n>n_j$. By taking the imaginary part of both sides of (3.21) one finds

$$\lim_{n \to \infty} \Bigg| \operatorname{sign}[T'(x)] \sqrt{1-T^2(x)}\, p_{nN+j}(x) - |\varphi_j(x)| \left(\prod_{i=1}^{n_j} |t_{iN+j}| \right) \sin \left(\sum_{i=1}^{n+1} \arg t_{iN+j} + \arg \varphi_j(x) \right) \Bigg| = 0 \tag{3.27}$$

uniformly on K

THEOREM 5. *Suppose that* (2.2) *and* (3.1) *hold; then the orthogonality measure μ is absolutely continuous in E. If $d\mu(x) = w(x)\, dx$ on E and if K is a closed set in $E \backslash \{\omega^{2N}(x)=1\}$, then*

$$\left(\prod_{i=1}^{\infty} |t_{iN+j}|^2 \right) |\varphi_j(x)|^2 = \frac{|q_{N-1}^{(j+1)}(x)| \sqrt{1-T^2(x)}}{\pi a_{j+1}^0 w(x)} \qquad x \in K. \tag{3.28}$$

Proof. A simple computation gives for $x \in K$ and for n large enough

$$|t_{nN+j} p_{nN+j}(x) - p_{(n-1)N+j}(x)|^2 = p_{nN+j}^2(x) + p_{(n-1)N+j}^2(x) - \left(p_N^{(nN+j)}(x) - \frac{a_{(n+1)N+j}}{a_{nN+j+1}} p_{N-2}^{(nN+j+1)}(x) \right) p_{nN+j}(x)\, p_{(n-1)N+j}(x);$$

hence, by Lemma 1, we have for every continuous function f with support in K

$$\lim_{n \to \infty} \int_E f(x)\, |t_{nN+j} p_{nN+j}(x) - p_{(n-1)N+j}(x)|^2\, d\mu(x) = \frac{1}{\pi a_{j+1}^0} \int_E f(x)\, |q_{N-1}^{(j+1)}(x)| \sqrt{1-T^2(x)}\, dx.$$

We have used the fact that $q_{2N-1}^{(j+1)}(x)=2T(x)q_{N-1}^{(j+1)}(x)$ in the previous calculation. On the other hand, by the previous theorem we also have

$$\lim_{n\to\infty}\int_E f(x)\,|t_{nN+j}p_{nN+j}(x)-p_{(n-1)N+j}(x)|^2\,d\mu(x)$$
$$=\int_E f(x)\left(\prod_{i=1}^{\infty}|t_{iN+j}|^2\right)|\varphi_j(x)|^2\,d\mu(x).$$

Note that the product in the latter expression contains only a finite number of factors because when $x\in K$ one has $|t_{nN+j}|=1$ for $n>n_j$. Comparing both equations we find that

$$\left(\prod_{i=1}^{\infty}|t_{iN+j}|^2\right)|\varphi_j(x)|^2\,d\mu(x)=\frac{1}{\pi a_{j+1}^0}\,|q_{N-1}^{(j+1)}(x)|\,\sqrt{1-T^2(x)}\,dx. \tag{3.29}$$

It follows that if $x_0\in E\setminus\{\omega^{2N}(x)=1\}$ is a mass point of μ, then $\varphi_j(x_0)=0$. It is possible to show from (3.24) and (3.20) that φ_j cannot become zero at an interior point of E; the proof of this is along exactly the same lines as for the case $N=1$ given in Máté and Nevai [14] and hence we do not give the details here. Therefore μ is absolutely continuous in $E\setminus\{\omega^{2N}(x)=1\}$, and (3.28) then follows from (3.29). ∎

4. Shifted Turán Determinants

Our next aim is to generalize Theorem 1 for several intervals. This means that we need a proper generalization of the Turán determinant (1.4). We define the *shifted Turán determinant* by

$$D_n(x;N)=p_n(x)\,p_{n-N+1}(x)-\frac{a_{n+1}}{a_{n-N+1}}\,p_{n+1}(x)\,p_{n-N}(x)$$
$$=\frac{1}{a_{n-N+1}}\begin{vmatrix} p_n(x) & a_{n+1}\,p_{n+1}(x)\\ p_{n-N}(x) & a_{n-N+1}\,p_{n-N+1}(x)\end{vmatrix}. \tag{4.1}$$

In the definition of the ordinary Turán determinant (1.4) we see that the bottom line in the determinant is the same as the top line except that n has changed to $n-1$. For the shifted Turán determinant given by (4.1) a similar thing is true but now n has changed to $n-N$.

These shifted Turán determinants can be computed recursively and they are given in terms of the differences b_k-b_{k-N} and a_k-a_{k-N} very much as the ordinary Turán determinants (see, e.g., [4] and references there):

LEMMA 6. *Shifted Turán determinants always satisfy*

$$D_n(x;N)=\left(\prod_{j=N}^{n}\frac{a_j}{a_{j-N+1}}\right)\left\{p_{N-1}(x)+\sum_{k=N}^{n}\left(\prod_{j=N}^{k}\frac{a_{j-N+1}}{a_j}\right)\right.$$
$$\left.\times\left[\frac{b_k-b_{k-N}}{a_{k-N+1}}\,p_{k-N}(x)+\frac{a_k^2-a_{k-N}^2}{a_{k-N+1}a_{k-N}}\,p_{k-N-1}(x)\right]p_k(x)\right\}. \tag{4.2}$$

Proof. Use the recurrence formula (1.2) for $a_{n+1}p_{n+1}(x)$ in the definition (4.1); then

$$D_n(x;N)=p_n(x)\,p_{n-N+1}(x)-\frac{1}{a_{n-N+1}}\,((x-b_n)\,p_n(x)-a_n\,p_{n-1}(x))\,p_{n-N}(x)$$
$$=\frac{a_n}{a_{n-N+1}}\,D_{n-1}(x;N)+\frac{p_n(x)}{a_{n-N+1}}$$
$$\times\left(a_{n-N+1}\,p_{n-N+1}(x)+\frac{a_n^2}{a_{n-N}}\,p_{n-N-1}(x)-(x-b_n)\,p_{n-N}(x)\right).$$

Now use the recurrence relation (1.2) for $a_{n-N+1}\,p_{n-N+1}(x)$; then

$$D_n(x;N)=\frac{a_n}{a_{n-N+1}}\,D_{n-1}(x;N)$$
$$+\left[\frac{b_n-b_{n-N}}{a_{n-N+1}}\,p_{n-N}(x)+\frac{a_n^2-a_{n-N}^2}{a_{n-N+1}a_{n-N}}\,p_{n-N-1}(x)\right]p_n(x).$$

The result now follows immediately by iteration. ∎

We are now ready to formulate and prove a generalization of Theorem 1:

THEOREM 6. *Suppose that* (2.2) *holds and that the recurrence coefficients are of bounded variation* (mod N), *i.e.*,

$$\sum_{k=0}^{\infty}\left(|b_k-b_{k+N}|+|a_{k+1}-a_{k+N+1}|\right)<\infty; \tag{4.3}$$

then μ is absolutely continuous in $E\setminus\{\omega^{2N}(x)=1\}$, $w(x)=\mu'(x)$ is strictly positive and continuous on $E\setminus\{\omega^{2N}(x)=1\}$, and

$$\lim_{n\to\infty}D_{nN+j}(x;N)=\frac{1}{\pi a_{j+1}^0}\,\frac{\sqrt{1-T^2(x)}}{w(x)}\,\mathrm{sign}[T'(x)] \tag{4.4}$$

holds uniformly on every closed subset of $E \setminus \{\omega^{2N}(x) = 1\}$. *An upper bound for the error of approximation for* x *on a closed subset of* $E \setminus \{\omega^{2N}(x) = 1\}$ *is given by*

$$\left| a_{n+1}^0 D_n(x; N) - \frac{1}{\pi} \frac{\sqrt{1 - T^2(x)}}{w(x)} \operatorname{sign}[T'(x)] \right| \leqslant C \sum_{k=n+1}^{\infty} \left(\left| \frac{b_k - b_{k-N}}{a_{k-N+1}} \right| + \left| \frac{a_k^2 - a_{k-N}^2}{a_{k-N+1} a_{k-N}} \right| \right), \tag{4.5}$$

where C *is a constant depending on the closed subset.*

Proof. We adapt the methods of [25, Sect. 3] to the present situation. The absolute continuity follows from Theorem 5. If K is a closed set in $E \setminus \{\omega^{2N}(x) = 1\}$, then for $x \in K$ there exists an integer n_j such that $|t_{nN+j}| = 1$ for $n > n_j$. Hence the infinite series

$$\psi(x) = p_{N-1}(x) + \sum_{k=N}^{\infty} \left(\prod_{j=N}^{k} \frac{a_{j-N+1}}{a_j} \right) \left[\frac{b_k - b_{k-N}}{a_{k-N+1}} p_{k-N}(x) + \frac{a_k^2 - a_{k-N}^2}{a_{k-N+1} a_{k-N}} p_{k-N-1}(x) \right] p_k(x) \tag{4.6}$$

converges uniformly for $x \in K$ and, by Lemma 5,

$$\lim_{n \to \infty} D_{nN+j}(x; N) = \left(\prod_{i=1}^{N} \frac{a_i^0}{a_i} \right) \frac{a_N^0}{a_{j+1}^0} \psi(x)$$

holds uniformly for $x \in K$. From Lemma 1 we find for every continuous function f with support in K

$$\lim_{n \to \infty} \int_E f(x) D_{nN+j}(x; N) w(x)\, dx = \frac{1}{4\pi} \frac{1}{a_{j+1}^0} \int_E f(x) \frac{2 - q_{2N}^{(j+1)}(x) + (a_{j+1}^0 / a_{j+2}^0)\, q_{2N-2}^{(j+2)}(x)}{\sqrt{1 - T^2(x)}} \operatorname{sign}[T'(x)]\, dx.$$

Use $q_{n+2N}^{(k)}(x) = 2T(x) q_{n+N}^{(k)}(x) - q_n^{(k)}(x)$ and (3.5) to obtain

$$\lim_{n \to \infty} \int_E f(x) D_{nN+j}(x; N) w(x)\, dx = \frac{1}{\pi} \frac{1}{a_{j+1}^0} \int_E f(x) \sqrt{1 - T^2(x)} \operatorname{sign}[T'(x)]\, dx.$$

On the other hand we have

$$\lim_{n\to\infty}\int_E f(x)D_{nN+j}(x;N)w(x)\,dx=\left(\prod_{i=1}^{N}\frac{a_i^0}{a_i}\right)\frac{a_N^0}{a_{j+1}^0}\int_E f(x)\psi(x)w(x)\,dx.$$

This means that for $x\in E\setminus\{\omega^{2N}(x)=1\}$

$$\left(\prod_{i=1}^{N}\frac{a_i^0}{a_i}\right)a_N^0\psi(x)w(x)=\frac{1}{\pi}\sqrt{1-T^2(x)}\,\mathrm{sign}[T'(x)],$$

from which (4.4) follows. The upper bound follows immediately from the upper bound (3.20) and Lemma 6. ∎

5. Examples

In this section we compare two approximations to the *sieved Legendre weight* on $[-1,1]$

$$w_N(x)=\tfrac{1}{2}\,|U_{N-1}(x)|\;(-1\leqslant x\leqslant 1), \tag{5.1}$$

where $U_{N-1}(x)$ is the *Chebyshev polynomial of the second kind* of degree $N-1$. The orthogonal polynomials corresponding to this weight are a special case of sieved ultraspherical polynomials (Al-Salam, Allaway, and Askey [1]). This weight is obtained from the Legendre weight $w_1(x)=\frac{1}{2}$ on $[-1,1]$ by a polynomial transformation (Badkov [3], Geronimo and Van Assche [8, Sect. VI, p. 578]) and therefore the recurrence coefficients can be obtained in terms of the recurrence coefficients of Legendre polynomials (in which case we have $a_n=n/\sqrt{4n^2-1}$ and $b_n=0$), giving

$$b_n=0\quad a_{nN+j}=\tfrac{1}{2}\qquad (j=2,\ldots,N-1)\quad (n=0,1,\ldots)$$

$$a_{nN}^2=\frac{1}{2}\frac{n}{2n+1},\quad a_{nN+1}^2=\frac{1}{2}\frac{n+1}{2n+1},\qquad (n=0,1,\ldots) \tag{5.2}$$

[8, Theorem 12, p. 578]. Note that $a_n\to\frac{1}{2}$ which means $q_n^{(k)}(x)=U_n(x)$ for every $k\geqslant 0$.

In Fig. 1 we have plotted the approximation

$$w_4(x)\approx\frac{2}{\pi}\frac{a_n\sqrt{1-x^2}}{a_n p_n^2(x)-a_{n+1}p_{n+1}(x)p_{n-1}(x)}$$

with $n=20$, using the ordinary Turán determinant $D_n(x)$. Note that from (5.2) one easily finds

$$a_{nN+1}^2-\frac{1}{4}=\frac{1}{8n+4}=\frac{1}{4}-a_{nN}^2$$

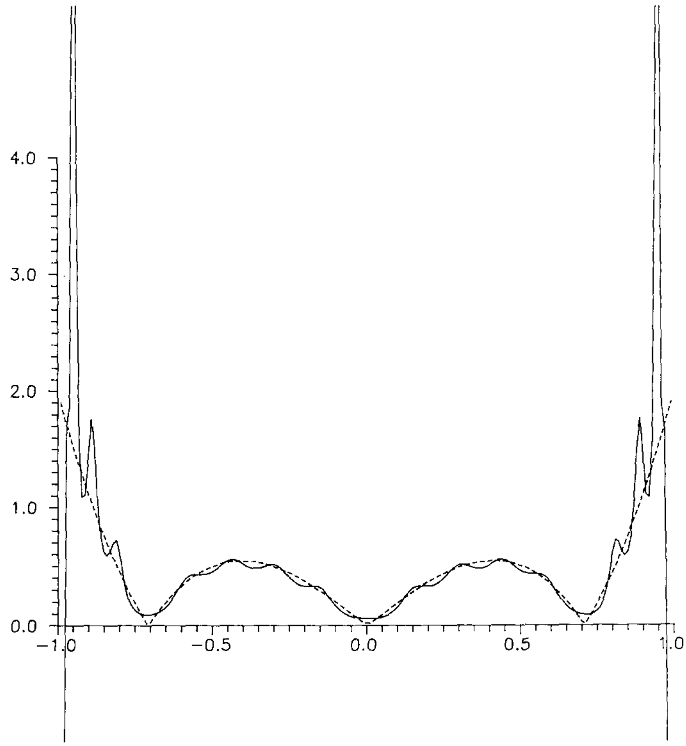

FIG. 1. Approximation with the ordinary Turán determinant.

which means that (1.6) is false and hence that Theorem 1 cannot be applied. Nevertheless these Turán determinants converge weakly to the weight in the sense that

$$\lim_{n \to \infty} \int_{-1}^{1} f(x) D_n(x) w_N(x)\, dx = \frac{2}{\pi} \int_{-1}^{1} f(x) \sqrt{1-x^2}\, dx$$

[25, Theorem 9, p. 449] and one also has

$$\lim_{n \to \infty} \int_{-1}^{1} \left| D_n(x) w_N(x) - \frac{2}{\pi} \sqrt{1-x^2} \right| dx = 0$$

[16, Theorem 10.1, p. 268]. Observe how the approximation oscillates around the weight function.

In Fig. 2 we have the approximation with the shifted Turán determinant

$$w_N(x) \approx \frac{2}{\pi} \frac{a_{n-N+1} U_{N-1}(x) \sqrt{1-x^2}}{a_{n-N+1} p_n(x) p_{n-N+1}(x) - a_{n+1} p_{n+1}(x) p_{n-N}(x)}$$

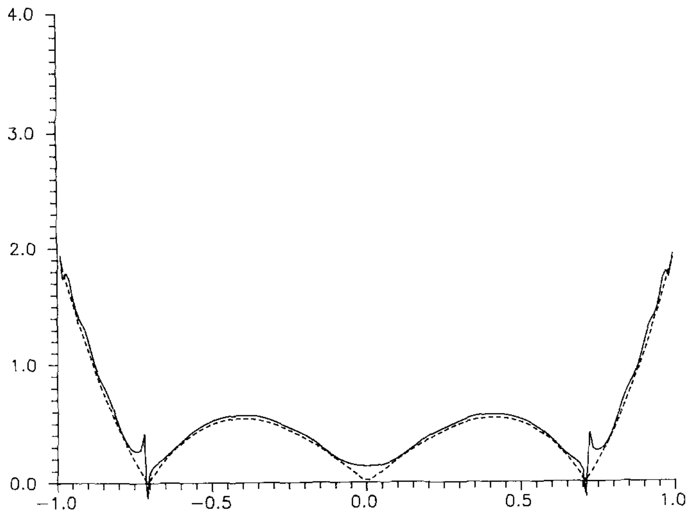

FIG. 2. Approximation with a shifted Turán determinant.

with $N=4$ and $n=20$. Note that this approximation does not oscillate around the weight but imitates the shape of weight very well, except near the zeros of the weight.

This is to be expected because from (5.2) we find

$$a_n - a_{n+N} = 0 \qquad [n \equiv j \pmod N),\ j=2, ..., N-1],$$
$$a^2_{(n+1)N} - a^2_{nN} = a^2_{nN+1} - a^2_{(n+1)N+1} = \frac{1}{2}\frac{1}{(2n+1)(2n+3)} \qquad (n=0, 1, ...), \tag{5.3}$$

which means that Theorem 6 can be applied. But this only gives uniform convergence on closed sets of $E \backslash \{\omega^{2N}(x)=1\}$, and in this case $E=[-1, 1]$ and the set $\{\omega^{2N}(x)=1\}$ coincides with the zeros of $U_{N-1}(x)$ and the points ± 1. The N intervals in this case are touching at the zeros of $U_{N-1}(x)$ and thus we have uniform convergence away from these zeros and away from ± 1. Actually the shifted Turán determinants will have zeros in the neighborhood of the zeros of $U_{N-1}(x)$ which causes the appearance of poles in the approximant. Note that from (1.8) and (5.3) we obtain an error of approximation of the order $O(1/n)$.

The approximations using (shifted) Turán determinants are easy to compute if one knows the recurrence coefficients of the orthogonal polynomials under consideration. The computation of the polynomials can then be done recursively. This procedure is known to be quite robust to the effect of rounding errors (see, e.g., Gautschi [5]) and allows us to compute Turán determinants up to very high order.

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